

A VARIANT PROOF OF  $\text{Con}(\mathfrak{b} < \mathfrak{a})$ 

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**Abstract.** We present a variation of the proof in [2] of  $\text{Con}(\mathfrak{b} < \mathfrak{a})$ , which in particular removes some of the obstacles to generalising the argument to cardinals  $\kappa > \omega$ .

**§1. Introduction.** The generalisations of cardinal characteristics of the continuum to cardinals  $\kappa$  greater than  $\omega$  has generated significant interest recently. A particular result that has so far resisted attempts at generalisation is the statement that  $\mathfrak{b} < \mathfrak{a}$  is consistent. Blass, Hyttinen and Zhang [1, Section 5] briefly survey the different approaches known for proving  $\text{Con}(\mathfrak{b} < \mathfrak{a})$ , highlighting the difficulties each presents for a generalisation.

We present here a variation on the proof of  $\text{Con}(\mathfrak{b} < \mathfrak{a})$  given in [2], which we hope will be more amenable to generalisation. In particular, the proof in [2] relies on a rank argument, which of course cannot be naïvely generalised to uncountable  $\kappa$ . We show here that it may be replaced by a suitable formulation in terms of games, which *does* generalise to higher  $\kappa$ . Indeed, with this observation, the question of forcing  $\mathfrak{b}_\kappa > \mathfrak{a}_\kappa$  for some suitable large cardinal  $\kappa$  seems to boil down to interesting questions about the existence of suitable filters on  $\kappa$ .

**§2. Preliminaries.** Let  $\kappa$  be an infinite cardinal. A family  $\mathcal{A} \subseteq [\kappa]^\kappa$  is called *almost disjoint* if  $|A \cap B| < \kappa$  for any two distinct members  $A$  and  $B$  of  $\mathcal{A}$ .  $\mathcal{A}$  is a *maximal almost disjoint family* (*mad family*, for short) if  $\mathcal{A}$  is almost disjoint and maximal with this property. This means that for every  $C \in [\kappa]^\kappa$  there is  $A \in \mathcal{A}$  such that  $|A \cap C| = \kappa$ . The *almost disjointness number*  $\mathfrak{a}_\kappa$  is the least

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size of a mad family on  $\kappa$  of size at least  $cf(\kappa)$  (equivalently, of size  $> cf(\kappa)$ ). In case  $\kappa = \omega$  write  $\mathfrak{a}$  for  $\mathfrak{a}_\omega$ .

Now assume  $\kappa$  is a regular cardinal. For functions  $f, g \in \kappa^\kappa$ , say that  $g$  *eventually dominates*  $f$  ( $f \leq^* g$  in symbols) if  $f(\alpha) \leq g(\alpha)$  holds for all  $\alpha$  beyond some  $\alpha_0 < \kappa$ . The *unbounding number*  $\mathfrak{b}_\kappa$  is the least size of an unbounded family  $\mathcal{F}$  in the order  $(\kappa^\kappa, \leq^*)$ . That is, for all  $g \in \kappa^\kappa$  there is  $f \in \mathcal{F}$  with  $f(\alpha) > g(\alpha)$  for cofinally many  $\alpha$ 's. Again we write  $\mathfrak{b}$  instead of  $\mathfrak{b}_\omega$ .

Let  $\mathcal{F}$  be a filter on  $\omega$ . *Mathias forcing*  $\mathbb{M}(\mathcal{F})$  with  $\mathcal{F}$  consists of conditions  $(s, F)$  such that  $s \in [\omega]^{<\omega}$ ,  $F \in \mathcal{F}$ , and  $\max(s) < \min(F)$ .  $\mathbb{M}(\mathcal{F})$  is ordered by  $(t, G) \leq (s, F)$  if  $s \subseteq t \subseteq s \cup F$  and  $G \subseteq F$ . It is well-known and easy to see that  $\mathbb{M}(\mathcal{F})$  is a  $\sigma$ -centered forcing which introduces a pseudointersection  $Z$  of the filter  $\mathcal{F}$ . This means that  $Z \subseteq^* F$  for all  $F \in \mathcal{F}$ , where  $\subseteq^*$  denotes *almost inclusion*:  $A \subseteq^* B$  iff  $A \setminus B$  is finite.

In [2], the notion of *pseudocontinuity* is used. This notion and the corresponding basic lemma can be nicely phrased in terms of continuity with respect to an appropriate topology.

**DEFINITION 1.** *The initial segment topology on  $\omega$  is the topology which has the (von Neumann) ordinals as open sets. We denote  $\omega$  endowed with this topology by  $\omega_i$ .*

**DEFINITION 2.** *A function to  $\omega$  or  $\omega^\omega$  is pseudocontinuous if it is continuous as a function to  $\omega_i$  or  $\omega_i^\omega$  respectively.*

Thus, a pseudocontinuous function  $F : X \rightarrow \omega$  is one such that for every  $n \in \omega$ , the set of  $x$  in  $X$  with image at most  $n$  is open.

**LEMMA 3.** *Compact sets in  $\omega_i$  and  $\omega_i^\omega$  are bounded. In particular, any pseudocontinuous image in  $\omega$  or  $\omega^\omega$  of a compact set must be bounded.*

**PROOF.** The Lemma is clear for  $\omega_i$ . Similarly, compact  $K \subset \omega_i^\omega$  are in fact bounded in the strict (not just  $\leq^*$ ) sense. Otherwise, there would be some  $m$  in  $\omega$  such that  $f(m)$  is unbounded in  $\omega$  for  $f \in K$ , and then the open sets  $\mathcal{O}_{m,n} = \{f \in \omega_i^\omega \mid f(m) \leq n\}$  for  $n < \omega$  would form an open cover of  $K$  with no finite subcover.  $\dashv$

As usual we may identify  $\mathcal{P}(\omega)$  with  $2^\omega$  by way of the map taking sets to their characteristic functions,  $\chi : X \mapsto \chi_X$ . We give  $\mathcal{P}(\omega)$  the corresponding topology, making  $\chi$  a homeomorphism from  $\mathcal{P}(\omega)$  to the Cantor space  $2^\omega$ .

DEFINITION 4. For any cardinal  $\lambda$ , we call a filter  $\mathcal{G} \subseteq \mathcal{P}(\omega)$  a  $K_\lambda$ -filter if it is generated by the union of fewer than  $\lambda$  many compact subsets of  $\mathcal{P}(\omega)$ . We write  $K_\sigma$  for  $K_{\aleph_1}$ .

LEMMA 5. If  $K_0, \dots, K_{n-1}$  are (finitely many) compact subsets of  $\mathcal{P}(\omega)$ , then the pointwise intersection

$$\bigwedge_{i < n} K_i = \left\{ \bigcap_{i < n} G_i \mid (G_0, \dots, G_{n-1}) \in \prod_{i < n} K_i \right\}$$

and the pointwise union

$$\bigvee_{i < n} K_i = \left\{ \bigcup_{i < n} G_i \mid (G_0, \dots, G_{n-1}) \in \prod_{i < n} K_i \right\}$$

are compact. Furthermore, for any compact set  $K \subseteq \mathcal{P}(\omega)$ , the upward closure

$$\bar{K} = \{A \in \mathcal{P}(\omega) \mid \exists B \in K (A \supseteq B)\}$$

is also compact.

PROOF. The product  $\prod_{i < n} K_i$  is compact by the Tychonoff theorem, and the functions  $\mathcal{P}(\omega)^n \rightarrow \mathcal{P}(\omega)$  given by  $(G_0, \dots, G_{n-1}) \mapsto \bigcap_{i < n} G_i$  and  $(G_0, \dots, G_{n-1}) \mapsto \bigcup_{i < n} G_i$  are clearly continuous, so  $\bigwedge_{i < n} K_i$  and  $\bigvee_{i < n} K_i$  are compact. Finally, for compact  $K \subseteq \mathcal{P}(\omega)$ ,  $\bar{K}$  is just  $K \vee \mathcal{P}(\omega)$ .  $\dashv$

**§3. The proof.** We work in a model  $V$  of ZFC in which  $\lambda = \mathfrak{c}^V$  is a regular cardinal satisfying  $2^\lambda = \lambda^+$ , and there is an unbounded,  $<^*$ -well-ordered sequence  $\langle f_\alpha : \alpha < \lambda \rangle$  of strictly increasing functions from  $\omega$  to  $\omega$ . For example, any model of GCH will suffice as a ground model, and these properties will be preserved in intermediate stages of our forcing iteration.

Let  $\mathcal{A}$  be an infinite maximal almost disjoint family in  $V$  of subsets of  $\omega$ .

THEOREM 6. There is a ccc forcing  $\mathbb{P}(\mathcal{A})$  such that

$$\Vdash_{\mathbb{P}(\mathcal{A})} \mathcal{A} \text{ is not mad and } \langle f_\alpha : \alpha < \lambda \rangle \text{ is still unbounded.}$$

PROOF. Let  $\mathcal{F} = \mathcal{F}(\mathcal{A})$  be the dual filter of  $\mathcal{A}$ , that is, the filter generated by the sets whose complements are finite or in  $\mathcal{A}$ . Note that this filter is proper: if for some  $k < \omega$  there were  $\{A_i \mid i < k\} \subset \mathcal{A}$  such that  $|\bigcap_{i < k} \omega \setminus A_i| < \omega$ , any other element of  $\mathcal{A}$  would have infinite intersection with one of the  $A_i$ , violating almost disjointness. Note that the generic subset of  $\omega$  introduced by Mathias forcing with

$\mathcal{F}$ , or any filter extending  $\mathcal{F}$ , will end the madness of  $\mathcal{A}$ , as it will be almost contained in  $\omega \setminus A$  for every  $A \in \mathcal{A}$ .

First we add  $\lambda$  many Cohen reals. It is well-known that the unboundedness of  $\langle f_\alpha : \alpha < \lambda \rangle$  is preserved in this intermediate extension. In case  $\mathcal{A}$  is not mad anymore in this extension we are done. Also, if  $\mathcal{F}$  is contained in a  $K_\lambda$  filter  $\mathcal{G}$  in the intermediate extension, we may simply force with  $\mathbb{M}(\mathcal{G})$  for it is well-known, and easy to see [2, 3.2], that Mathias forcing with a  $K_\lambda$ -filter does not destroy the unboundedness of  $\langle f_\alpha : \alpha < \lambda \rangle$ . So assume that  $\mathcal{F}$  is not contained in any  $K_\lambda$ -filter.

We shall recursively construct a filter  $\mathcal{G} \supseteq \mathcal{F}$  such that furthermore

$$(*) \quad \Vdash_{\mathbb{M}(\mathcal{G})} \langle f_\alpha : \alpha < \lambda \rangle \text{ is unbounded.}$$

Along the construction we shall take care of every potential  $\mathbb{M}(\mathcal{G})$ -name for a function in  $\omega^\omega$ , either “killing it” or “sealing it off”.

To be precise: let us refer to partial functions  $\tau : [\omega]^{<\omega} \times \omega \dashrightarrow \omega$  as *preterms*, and let  $\mathcal{T} = \{\tau_\beta : \beta < \lambda\}$  be an enumeration of the set of all preterms. Note in particular that if  $\mathcal{G} \supseteq \mathcal{F}$  is a filter and  $\dot{g}$  is an  $\mathbb{M}(\mathcal{G})$ -name for a function in  $\omega^\omega$ , then  $\tau = \tau_{\dot{g}}$  given by

$$\tau(s, m) = n \text{ iff } \exists G \in \mathcal{G} ((s, G) \Vdash \dot{g}(m) = n)$$

is a preterm, the *preterm associated with  $\dot{g}$* . We shall constrain attention to names  $\dot{g}$  such that  $\mathbb{1} \Vdash_{\mathbb{M}(\mathcal{G})} \dot{g} \in \omega^\omega$ , since every function from  $\omega$  to  $\omega$  in the generic extension has such a name; we call such names *total names*.

We construct filters  $\mathcal{G}_\beta$  for  $0 \leq \beta \leq \lambda$ , starting from  $\mathcal{G}_0 = \mathcal{F}$ , such that

- for each  $\beta < \lambda$ ,  $\mathcal{G}_{\beta+1}$  is generated by  $\mathcal{G}_\beta$  and a  $K_\sigma$  filter  $\mathcal{H}_\beta$ ,
- $\mathcal{G}_\delta = \bigcup_{\beta < \delta} \mathcal{G}_\beta$  for each limit ordinal  $\delta \leq \lambda$ ,

and either

**(KILL):** for all filters  $\mathcal{H} \supseteq \mathcal{G}_{\beta+1}$ ,  $\tau_\beta$  is not associated with any total  $\mathbb{M}(\mathcal{H})$ -name, or

**(SEAL):** there is an  $\alpha < \lambda$  such that for all filters  $\mathcal{H} \supseteq \mathcal{G}_{\beta+1}$  and all  $\mathbb{M}(\mathcal{H})$ -names  $\dot{g}$ , if  $\tau_{\dot{g}} = \tau_\beta$  then  $\Vdash_{\mathbb{M}(\mathcal{H})} \dot{g} \not\leq^* \check{f}_\alpha$ .

Clearly any filter  $\mathcal{G} \supseteq \mathcal{G}_\lambda$  will then satisfy (\*).

So suppose  $\mathcal{G}_\beta$  has been defined for some  $\beta < \lambda$ ; we wish to find an appropriate  $K_\sigma$  filter  $\mathcal{H}_\beta$ . Note that  $\mathcal{G}_\beta$  is generated by  $\mathcal{F}$  and a  $K_\lambda$  filter  $\mathcal{G}'_\beta$ ; without loss of generality we may assume that  $\mathcal{F}$  contains all cofinite subsets of  $\omega$ . Let  $\mathcal{K}_\beta$  be a family of fewer than  $\lambda$  many compact subsets of  $2^\omega$  generating  $\mathcal{G}'_\beta$ . By Lemma 5, we may assume

that  $\mathcal{K}_\beta$  is closed under finite pointwise intersections, and that for all  $K \in \mathcal{K}_\beta$ ,  $K$  is upwards-closed under  $\subseteq$ , so that  $\mathcal{G}'_\beta = \bigcup \mathcal{K}_\beta$ .

Everything that has come so far can actually be considered to have occurred in a partial extension model, between the original model and the full extension with  $\lambda$ -many Cohens. More explicitly, all (codes of) elements of  $\mathcal{K}_\beta$  belong to this intermediate model.

Let  $\subset_{\text{ee}}$  denote the strict end-extension relation on  $[\omega]^{<\omega}$ : that is,  $s \subset_{\text{ee}} s'$  if and only if  $s \subset s'$  and  $\max(s) < \min(s' \setminus s)$ ; define  $\subseteq_{\text{ee}}$ ,  $\supset_{\text{ee}}$  and  $\supseteq_{\text{ee}}$  accordingly.

In [2], a rank function was used. For our generalisation, we take a different approach using games, but use these games to much the same end as the rank function is used in [2]. It should be noted that our games are very closely related to the games independently introduced by Guzmán, Hrušák, and Martínez [3], also in the context of a proof of  $\text{Con}(\mathfrak{b} < \mathfrak{a})$ .

Let  $\tau = \tau_\beta$ .

**DEFINITION 7.** *Given  $\tau \in \mathcal{T}$ , the  $\tau$  nominalisation exercise is the following game. There are two players, Sensei and Student. On turn 0, Sensei chooses an  $m \in \omega$  and  $t_0 \in [\omega]^{<\omega}$ . At odd stages  $2d + 1$ , Student plays a filter set  $F(d) \in \mathcal{F}$  and a compact set  $K(d) \in \mathcal{K}_\beta$ . At even stages  $2d + 2$ , Sensei plays an element  $t_{d+1}$  of  $[\omega]^{<\omega}$  such that*

- $t_{d+1}$  end-extends  $t_d$
- $t_{d+1} \setminus t_d \subseteq F(d)$
- $t_{d+1} \setminus t_d$  meets every member of  $K(d)$ .

*If there is  $s \subseteq t_{d+1}$  end extending  $t_0$  such that  $(s, m) \in \text{dom}(\tau)$ , Sensei declares Student to have passed and the game ends. If the game continues for infinitely many stages, then (clearly) Student has failed.*

Note that, since  $\mathcal{G}_\beta$  is a filter, and by compactness of  $K(d)$ , a  $t_{d+1}$  satisfying the requirements always exists. Also notice that if Student wins, he wins after finitely many steps. Hence the game is open and, by the classical Gale-Stewart Theorem, determined.

As in [2], we now distinguish two cases (in [2] they are *Subcases*), corresponding to options (KILL) and (SEAL) above.

**3.1. Case a.** There are  $m \in \omega$  and  $t_0 \in [\omega]^{<\omega}$  such that Sensei has a winning strategy in the  $\tau$  nominalisation exercise with 0th move  $(m, t_0)$ : play will continue for infinitely many steps. In this case we

shall choose  $\mathcal{H}_\beta$  in such a way that (KILL) holds:  $\tau$  will not correspond to a name for a function  $\omega \rightarrow \omega$  in the generic extension. The reader may wish to remember which case is which by the mnemonic “the  $\tau$  that can be named is not the eternal  $\tau$ .”

We shall actually work in the extension of such the intermediate model by one further Cohen function  $c : \omega \rightarrow \omega$ .

Consider the tree  $T$  of all possible sequences of plays  $(t_0, t_1, t_2, \dots)$  for Sensei according to his strategy, corresponding to all possible plays of Student. Note that  $T$  is infinitely branching since  $\mathcal{F}$  extends the Frechet filter. Use the Cohen function  $c$  to choose a branch through  $T$ , and denote the union of the  $t_i$  of this branch by  $G$ . There is no  $(s, m)$  with  $m$  from Sensei’s first move and  $t_0 \subseteq_{\text{ee}} s \subseteq G$  such that  $(s, m) \in \text{dom}(\tau_\beta)$ . Indeed otherwise, the  $\tau_\beta$  nominalisation exercise would have ended once Sensei played  $t_d$  sufficiently long to cover  $s$ . Thus, for any filter  $\mathcal{H} \ni G$ ,  $\tau \neq \tau_{\dot{g}}$  for any total  $\mathbb{M}(\mathcal{H})$  name  $\dot{g}$ . We may therefore simply take  $\mathcal{H}_\beta = \{G\}$  in order to satisfy (KILL). To check that  $\{G\} \cup \mathcal{G}_\beta$  generates a filter, consider any  $F \in \mathcal{F}$  and  $G' \in \mathcal{G}'_\beta$ , say  $G'$  is in the compact set  $K \in \mathcal{K}_\beta$ . For every  $t_d \in T$ , there is a successor node  $t_{d+1}$  in the tree  $T$  that is Sensei’s response, according to his strategy, to Student playing  $F$  and  $K$ , and so in particular this  $t_{d+1}$  meets the intersection of  $F$  and every member of  $K$ . Thus, by Cohen genericity we have that  $|G \cap F \cap G'| = \omega$ , completing Case a. (Note that  $G'$  may not belong to the intermediate model; this, however, is irrelevant for it is sufficient that  $K$  does. By genericity the Cohen real  $c$  will produce infinitely many  $d$  such that  $t_{d+1} \setminus t_d$  is contained in  $F$  and meets every  $G'' \in K$ , and this is clearly absolute and thus also holds for  $G'$ .)

**3.2. Case b.** The negation of Case a: for every 0th move  $(m, t_0)$  by Sensei, Student has a winning strategy in the  $\tau_\beta$  nominalisation exercise. In this case we wish to choose  $\mathcal{H}_\beta$  in such a way that (SEAL) holds.

Since Sensei chooses his moves from a countable set, there are clearly only countable many filter sets  $F_\ell \in \mathcal{F}$ ,  $\ell \in \omega$ , which appear as  $F(d)$  in some  $2d + 1$ st move of Student playing according to his strategy.

Suppose that for all but less than  $\lambda$  many members  $A$  of  $\mathcal{A}$ , there is  $G \in \mathcal{G}'_\beta$  such that  $A \cap G$  is finite. Then, adding less than  $\lambda$  many sets of the form  $\omega \setminus A$ ,  $A \in \mathcal{A}$ , to  $\mathcal{G}'_\beta$  results in a  $K_\lambda$  filter containing  $\mathcal{F}$ . This contradicts our initial assumption. Hence, for  $\lambda$  many  $A \in \mathcal{A}$ ,  $A \cap G$  is infinite for all  $G \in \mathcal{G}'_\beta$ . Let  $A_j$ ,  $j \in \omega$ , be countably many

such  $A$ 's such that for each  $j$  and  $\ell$ ,  $A_j$  is almost contained in  $F_\ell$ : this is possible because  $\mathcal{F}$  is the dual filter of the mad family  $\mathcal{A}$ .

For each  $G' \in \mathcal{G}'_\beta$ ,  $k \in \omega$ ,  $j \in \omega$ , and finite subset  $T$  of  $[\omega]^{<\omega}$ , we define a function  $f_{G',k,j,T} : \omega \rightarrow \omega$  as follows.

$$f_{G',k,j,T}(m) = \min\{n \mid \text{for any partition } A_j = \bigcup_{i < k} B_i \text{ there is } i < k \text{ s.t.} \\ \forall t \in T \exists s \supset_{\text{ee}} t (s \setminus t \subseteq B_i \cap G' \wedge \tau_\beta(s, m) \leq n)\}.$$

LEMMA 8. *For every  $G' \in \mathcal{G}'_\beta$ ,  $k, j \in \omega$ , and  $T \in [[\omega]^{<\omega}]^{<\omega}$ ,  $f_{G',k,j,T}$  is well-defined.*

PROOF. Fix  $m \in \omega$ . Given a partition  $\{B_i \mid i < k\}$  of  $A_j$ , let “ $n$  suffices for  $\{B_i \mid i < k\}$ ” mean the natural thing in the context of the definition of  $f_{G',k,j,T}$ , namely, that there is  $i < k$  such that for every  $t \in T$  there is  $s \supset_{\text{ee}} t$  with  $s \setminus t \subseteq B_i \cap G'$  and  $\tau_\beta(s, m) \leq n$ . So now fix a partition  $\{B_i \mid i < k\}$  of  $A_j$ ; we shall show that there is a  $n \in \omega$  that suffices for it. Let  $i < k$  be such that  $|B_i \cap G' \cap G| = \omega$  for every  $G \in \mathcal{G}'_\beta$ : such an  $i$  must exist, since  $A_j$  has infinite intersection with every member of the filter  $\mathcal{G}'_\beta$ . Finally, fix  $t \in T$ .

Consider a play of the  $\tau_\beta$  naming exercise in which Student follows his strategy, Sensei's 0th move is  $(m, t_0)$  with  $t_0 = t$ , and his later moves always satisfy the additional requirement  $t_{d+1} \setminus t_d \subseteq B_i \cap G'$ . Since  $B_i$  is almost contained in all  $F(d)$  played by Student according to his strategy and since  $B_i$  has infinite intersection with all  $G \in \mathcal{G}'_\beta$ , Sensei always has a valid such move.

So we have that eventually Sensei plays a  $t_d$  such that

$$\exists n_t \in \omega \exists s \subseteq t_d (s \supset_{\text{ee}} t \wedge \tau_\beta(s, m) = n_t).$$

Of course, by the construction of the game,  $s \setminus t_0 \subseteq B_i \cap G'$ . Taking such an  $n_t$  for each  $t \in T$  and setting  $n = \max_{t \in T}(n_t)$ , we have that  $n$  suffices for  $\{B_i \mid i < k\}$ .

Now, with  $k$  still fixed but allowing the partition  $\{B_i \mid i < k\}$  to vary, let us denote by  $n(\{B_i \mid i < k\})$  the least  $n$  that suffices for  $\{B_i \mid i < k\}$ . The space of partitions of  $A_j$  into  $k$  pieces can be identified with  $k^{A_j}$  and thus when endowed with the product topology is a compact topological space. Moreover, with this topology on the space of partitions, the function  $n$  sending  $\{B_i \mid i < k\}$  to  $n(\{B_i \mid i < k\})$  is clearly pseudocontinuous, since  $n$  being sufficient for  $\{B_i \mid i < k\}$  is witnessed by finitely many finite tuples  $s \setminus t$  from  $B_i$ , which of course define an open set in  $k^{A_j}$ . Thus by Lemma 3 the image of

the function  $n$  is bounded below  $\omega$ . The least such upper bound will be  $f_{G',k,j,T}(m)$ , and it follows that  $f_{G',k,j,T}$  is well-defined.  $\dashv$

LEMMA 9. *There exists an  $\alpha < \lambda$  such that for all  $G' \in \mathcal{G}'_\beta$ ,  $k, j \in \omega$  and  $T \in [[\omega]^{<\omega}]^{<\omega}$ ,  $f_\alpha \not\leq^* f_{G',k,j,T}$ .*

PROOF. We first note that, given  $k, j, T$ , and compact  $K \in \mathcal{K}_\beta$ , the function  $f_{\cdot,k,j,T}$  sending  $G'$  to  $f_{G',k,j,T}$  is pseudocontinuous from  $K$  to  $\omega^\omega$ , by much the same argument as in the proof of Lemma 8. Indeed, fixing  $m$  and  $n$ ,  $\{G' \mid f_{G',k,j,T}(m) \leq n\}$  is open in  $K$ .

We thus have from Lemma 3 that for each  $K \in \mathcal{K}_\beta$ ,  $f_{\cdot,k,j,T} \restriction K$  is bounded in  $\omega^\omega$ , say by  $h_K$ . Since  $\mathcal{K}_\beta$  has fewer than  $\lambda$  many elements, there is an  $\alpha < \lambda$  such that  $f_\alpha$  is not eventually dominated by any of the  $h_K$ , and hence not by any  $f_{G',k,j,T}$ .  $\dashv$

We now show that  $\alpha$  as given by Lemma 9 will make (SEAL) hold for an appropriate choice of  $\mathcal{H}_\beta$ . Given  $t \in [\omega]^{<\omega}$ ,  $G \in \mathcal{P}(\omega)$ , and  $m \in \omega$ , let

$$g_{t,G}^\beta(m) = \min\{n \mid \exists s \supseteq t (s \setminus t \subseteq G \wedge \tau_\beta(s, m) = n)\}$$

if the set on the right hand side is non-empty, and otherwise put  $g_{t,G}^\beta(m) = \omega$ . Thus,  $g_{t,G}^\beta$  is a function in  $(\omega + 1)^\omega$ . Let  $\alpha < \lambda$  be such that  $f_\alpha$  is not dominated by any  $f_{G',k,j,T}$ , as given by Lemma 9, and define

$$\mathcal{H}_\beta = \{H \subseteq \omega \mid \exists t \in [\omega]^{<\omega} (g_{t,\omega \setminus H}^\beta \geq^* f_\alpha)\}.$$

Note that given  $t \in [\omega]^{<\omega}$  and  $m_0 \in \omega$ , the set

$$\{H \subseteq \omega \mid \forall m \geq m_0 (g_{t,\omega \setminus H}^\beta(m) \geq f_\alpha(m))\}$$

is closed in  $\mathcal{P}(\omega)$ , and hence compact. Therefore,  $\mathcal{H}_\beta$  is a  $K_\sigma$  set.

To see that this set is an appropriate choice of  $\mathcal{H}_\beta$  as called for above, we check the following.

CLAIM 10. *Any filter  $\mathcal{H} \supseteq \mathcal{H}_\beta$  satisfies (SEAL).*

PROOF. Let  $\mathcal{H} \supseteq \mathcal{H}_\beta$  be a filter, and assume  $\tau_\beta = \tau_{\dot{g}}$  for some  $\mathbb{M}(\mathcal{H})$ -name  $\dot{g}$  for a function in  $\omega^\omega$ . Suppose there were  $(t, G) \in \mathbb{M}(\mathcal{H})$  and  $m_0 \in \omega$  such that

$$(t, G) \Vdash_{\mathbb{M}(\mathcal{H})} \forall m \geq \check{m}_0 (\dot{g}(m) \geq \check{f}_\alpha(m)).$$

By the definition of  $g_{t,G}^\beta$ , we must then also have  $g_{t,G}^\beta(m) \geq f_\alpha(m)$  for all  $m \geq m_0$ . So  $\omega \setminus G \in \mathcal{H}_\beta \subseteq \mathcal{H}$ , contradicting the fact that  $\mathcal{H}$  is a filter.  $\dashv$



CLAIM 11.  $\mathcal{H}_\beta \cup \mathcal{G}_\beta$  generates a filter.

PROOF. We take  $F \in \mathcal{F}$ ,  $G' \in \mathcal{G}'_\beta$ , and for some  $k < \omega$ ,  $H_i \in \mathcal{H}_\beta$  for  $i < k$ , and argue that  $F \cap G' \cap \bigcap_{i < k} H_i$  has cardinality  $\omega$ . Assume for the sake of contradiction that  $F \cap G' \subseteq^* \bigcup_{i < k} \omega \setminus H_i$ . For each  $i < k$ , fix  $t_i \in [\omega]^{<\omega}$  such that  $g_{t_i, \omega \setminus H_i}^\beta \geq^* f_\alpha$ . Also fix  $j$  such that  $A_j \subseteq^* F$ . Without loss of generality, we may take  $a < \omega$  such that  $A_j \setminus a \subseteq F$ ,  $F \cap G' \setminus a \subseteq \bigcup_{i < k} \omega \setminus H_i$  and  $\max(t_i) \geq a$  for every  $i < k$  (if necessary by extending each  $t_i$  with a sufficiently large element of  $\omega \setminus H_i$ : this can only increase the values of  $g_{t_i, \omega \setminus H_i}^\beta$ ). Fix  $m_0 \in \omega$  such that  $g_{t_i, \omega \setminus H_i}^\beta(m) \geq f_\alpha(m)$  for all  $m \geq m_0$  and  $i < k$ . Let  $T = \{t_i \mid i < k\}$  and let  $\{B_i \mid i < k\}$  be a partition of  $A_j$  such that  $B_i \cap G' \setminus a \subseteq \omega \setminus H_i$  for all  $i < k$ . By the definition of  $f_\alpha$ , there is some  $m > m_0$  such that  $f_\alpha(m) > f_{G', k, j, T}(m)$ ; take such a  $m$ , and denote  $f_{G', k, j, T}(m)$  by  $n$ . By the definition of  $f_{G', k, j, T}$ , there is an  $i$  such that for all  $t \in T$ , there is  $s \supset_{\text{ee}} t$  such that  $\tau_\beta(s, m) \leq n$  and  $s \setminus t$  is a subset of the intersection of  $G'$  and  $B_i$ . In particular,  $\min(s \setminus t_i) > \max(t_i) \geq a$ ,  $s \setminus t_i \subset B_i \cap G'$ , and  $\tau_\beta(s, m) \leq n$ . Thus  $s \setminus t_i \subseteq \omega \setminus H_i$ , from which we have  $g_{t_i, \omega \setminus H_i}^\beta(m) \leq n < f_\alpha(m)$ , contradicting the choice of  $m_0$ .  $\dashv$

This completes the construction of  $\mathcal{G}_{\beta+1}$  from  $\mathcal{G}_\beta$ , and hence the proof of Theorem 6.  $\dashv$

We are now ready for the consistency of  $\mathfrak{b} < \mathfrak{a}$ . Recall from the beginning of this section that our ground model  $V$  satisfies  $\mathfrak{c} = \lambda$  is regular,  $2^\lambda = \lambda^+$ , and  $\langle f_\alpha : \alpha < \lambda \rangle$  is unbounded  $<^*$ -well-ordered.

THEOREM 12. *There is a ccc forcing  $\mathbb{P}$  such that*

$$\Vdash_{\mathbb{P}} \mathfrak{a} = \lambda^+ \text{ and } \langle f_\alpha : \alpha < \lambda \rangle \text{ is still unbounded.}$$

*In particular,  $\mathfrak{b} \leq \lambda < \lambda^+ = \mathfrak{a}$  is consistent.*

PROOF. Perform a finite support iteration of orderings of type  $\mathbb{P}(\mathcal{A})$  of length  $\lambda^+$ , going through all (names for) mad families along the way by a bookkeeping argument (this is possible by the assumption  $2^\lambda = \lambda^+$ ). The unboundedness of  $\langle f_\alpha : \alpha < \lambda \rangle$  is preserved in the successor step of the iteration by Theorem 6 and in the limit step, by standard preservation results.  $\dashv$

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